

Distinguishing number and distinguishing index of neighbourhood corona of two graphs

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Abstract

The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has a vertex labeling (edge labeling) with d labels that is preserved only by a trivial automorphism. The neighbourhood corona of two graphs G_1 and G_2 is denoted by $G_1 \star G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 , and joining the neighbours of the i th vertex of G_1 to every vertex in the i th copy of G_2 . In this paper we describe the automorphisms of the graph $G_1 \star G_2$. Using results on automorphisms, we study the distinguishing number and the distinguishing index of $G_1 \star G_2$. We obtain upper bounds for $D(G_1 \star G_2)$ and $D'(G_1 \star G_2)$.

Keywords: Distinguishing index; Distinguishing number; neighborhood corona.

AMS Subj. Class.: 05C15, 05E18

1 Introduction

Let $G = (V, E)$ be a simple graph with n vertices. Throughout this paper we consider only simple graphs. The set of all automorphisms of G , with the operation of composition of permutations, is a permutation group on V and is denoted by $Aut(G)$. A labeling of G , $\phi : V \rightarrow \{1, 2, \dots, r\}$, is r -distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. In other words, ϕ is r -distinguishing if for every non-trivial $\sigma \in Aut(G)$, there exists x in V such that $\phi(x) \neq \phi(x\sigma)$. The distinguishing number of a graph G has defined by Albertson and Collins [1] and is the minimum number r such that G has a labeling that is r -distinguishing. Similar to this definition, Kalinkowski and Piłśniak [6] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. These indices has developed and number of papers published on this subject (see, for example [2, 7, 9]).

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We use the following notations: The set of vertices adjacent in G to a vertex of a vertex subset $W \subseteq V$ is the open neighborhood $N_G(W)$ of W . The closed neighborhood $G[W]$ also includes all vertices of W itself. In case of a singleton set $W = \{v\}$ we write $N_G(v)$ and $N_G[v]$ instead of $N_G(\{v\})$ and $N_G[\{v\}]$, respectively. We omit the subscript when the graph G is clear from the context. The complement of $N[v]$ in $V(G)$ is denoted by $\overline{N[v]}$. We denote the degree of a vertex v in graph G by $d_G(v)$ and the distance between two vertices u and w in graph G , by $dist_G(u, w)$. The corona of two graphs G and H which denoted by $G \circ H$ is defined in [4] and there have been some results on the corona of two graphs [3]. In [2] we have studied the distinguishing number and the distinguishing index of corona of two graphs. In this paper we consider another variation of corona of two graphs and study its distinguishing number and distinguishing index. Given simple graphs G_1 and G_2 , the neighbourhood corona of G_1 and G_2 , denoted by $G_1 \star G_2$ and is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the neighbours of the i th vertex of G_1 to every vertex in the i th copy of G_2 ([5]). Figure 1 shows $P_4 \star P_3$, where P_n is the path of order n . Liu and Zhu in [8] determined the adjacency spectrum of $G_1 \star G_2$ for arbitrary G_1 and G_2 and the Laplacian spectrum and signless Laplacian spectrum of $G_1 \star G_2$ for regular G_1 and arbitrary G_2 , in terms of the corresponding spectrum of G_1 and G_2 . Also Gopalapillai in [5] has studied the eigenvalues and spectrum of $G_1 \star G_2$, when G_2 is regular.

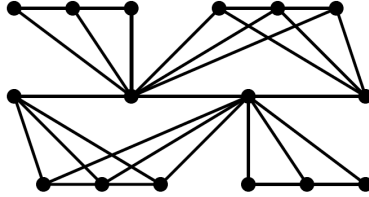


Figure 1: The neighbourhood corona of $P_4 \star P_3$.

In this paper we consider the neighbourhood corona of two graphs and discuss their distinguishing number and index. In the next section, we give a complete description of the automorphisms of neighbourhood corona of two arbitrary graphs. In Section 3, we study the distinguishing number and the distinguishing index of neighbourhood corona of two graphs.

2 Description of automorphisms of $G_1 \star G_2$

In this section we consider the neighbourhood corona of two graphs and describe its automorphisms. Let G_i has order n_i and size m_i ($i = 1, 2$). The neighbourhood corona $G_1 \star G_2$ of G_1 and G_2 has $n_1 + n_1 n_2$ vertices and $m_1(2n_2 + 1) + n_1 m_2$ edges and when $G_2 = K_1$, the graph $G_1 \star G_2$ is the splitting graph which has defined in [10].

Let $V(G_1) = \{v_1, v_2, \dots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \dots, u_{n_2}\}$. For $i = 1, 2, \dots, n_1$, let $u_1^i, u_2^i, \dots, u_{n_2}^i$ denote the vertices of the i th copy of G_2 , with the understanding

that u_j^i is the copy of u_j for each j . It is clear that the degrees of the vertices of $G_1 \star G_2$ are:

$$d_{G_1 \star G_2}(v_i) = (n_2 + 1)d_{G_1}(v_i), \quad i = 1, 2, \dots, n_1. \quad (1)$$

$$d_{G_1 \star G_2}(u_j^i) = d_{G_2}(u_j) + d_{G_1}(v_i), \quad i = 1, 2, \dots, n_1, \quad j = 1, 2, \dots, n_2. \quad (2)$$

Now we want to know how an automorphism of $G_1 \star G_2$ acts on the vertices G_1 and the vertices of copies G_2 . First we state and prove the following lemma.

Lemma 2.1 *Let G_1 and G_2 be two connected graphs such that $G_1 \neq K_1$ and f be an automorphism of $G_1 \star G_2$ such that $f(v_i) = u_j^k$ for some $i, k = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$. Then $d_{G_1}(v_k) > d_{G_1}(v_i)$.*

Proof. Since $f(v_i) = u_j^k$, so $d_{G_1 \star G_2}(v_i) = d_{G_1 \star G_2}(u_j^k)$. By Equations (1) and (2) we have $(n_2 + 1)d_{G_1}(v_i) = d_{G_2}(u_j) + d_{G_1}(v_k)$. By contradiction, suppose that $d_{G_1}(v_k) \leq d_{G_1}(v_i)$. Hence $(n_2 + 1)d_{G_1}(v_i) \leq d_{G_2}(u_j) + d_{G_1}(v_i)$, and so $n_2 d_{G_1}(v_i) \leq d_{G_2}(u_j)$. This contradiction forces us to conclude that $d_{G_1}(v_k) > d_{G_1}(v_i)$. \square

By Lemma 2.1 we can prove the following corollary:

Corollary 2.2 *Let G_1 be a connected graph such that $G_1 \neq K_1$ and f be an arbitrary automorphism of $G_1 \star G_2$.*

- (i) *If v is the vertex of G_1 with the maximum degree in G_1 , then $f(v) \in G_1$.*
- (ii) *If G_1 is a regular graph, then the restriction of f to G_1 is an automorphism of G_1 .*

We shall obtain some results for the automorphisms of $G_1 \star G_2$.

Lemma 2.3 *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2 , respectively, and $n_1 > 1$. Suppose that f is an automorphism of $G_1 \star G_2$ such that the restriction of f to G_1 is an automorphism of G_1 , and also f maps the copies of G_2 to each other. Then there are the automorphism g of G_1 and the automorphisms h_1, \dots, h_{n_1} of G_2 such that $f(G_2^i) = (h_i(G_2))^k$, where $v_k = g(v_i)$ and $i, k = 1, \dots, n_1$.*

Proof. Let f be an automorphism of $G_1 \star G_2$ such that the restriction of f to G_1 is an automorphism of G_1 , and also f maps the copies of G_2 to each other. Let f maps the i th copy of G_2 , G_2^i , to the j_i th copy of G_2 , $G_2^{j_i}$, where $i, j_i = 1, \dots, n_1$, such that for the fixed numbers i and j_i we have $f(u_k^i) = u_{k'}^{j_i}$, where $k, k' = 1, \dots, n_2$. Then we define the automorphism h_i on G_2 such that $h_i(u_k) = u_{k'}$. To complete the proof we need to show that the map g on $V(G_1)$ such that $g(v_i) = v_{j_i}$ is an automorphism of G_1 , where $i, j_i = 1, \dots, n_1$. Without loss of generality we can assume that the vertices v_1 and v_2 are adjacent, and show that v_{j_1} and v_{j_2} are adjacent. Since the vertices v_1 and v_2 are adjacent, the vertex v_1 is adjacent to each vertex of G_2^2 (we show this concept by $v_1 \sim G_2^2$). Hence $f(v_1) \sim (h_2(G_2))^{j_2}$, and so $f(v_1) \sim v_{j_2}$ and $v_1 \sim f^{-1}(v_{j_2})$, and thus $f^{-1}(v_{j_2}) \sim G_2^1$, and finally we have $v_{j_2} \sim G_2^{j_1}$. With a similar argument we can conclude that $f(v_2) \sim v_{j_1}$, and so $v_2 \sim f^{-1}(v_{j_1})$, and hence $f^{-1}(v_{j_1}) \sim G_2^2$, and thus

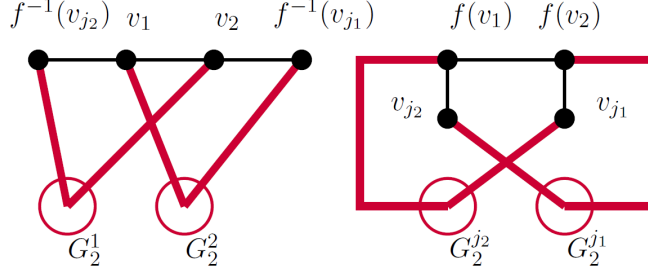


Figure 2: A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.3.

$v_{j_1} \sim G_2^{j_2}$ (see the Figure 2). On the other hand, since f maps G_2^1 to $(h_2(G_2))^{j_1}$, we have $d_{G_1 \star G_2}(u_k^1) = d_{G_1 \star G_2}((h_2(u_k))^{j_1})$. We deduce from Equations (1), (2) and $d_{G_2}(u_k) = d_{G_2}(h_2(u_k))$, that $d_{G_1}(v_1) = d_{G_1}(v_{j_1})$. Similarly, $d_{G_1}(v_2) = d_{G_1}(v_{j_2})$. Since the restriction of f to G_1 is an automorphism of G_1 , we have $d_{G_1}(v_1) = d_{G_1}(f(v_1))$ and $d_{G_1}(v_2) = d_{G_1}(f(v_2))$. Then

$$d_{G_1}(v_1) = d_{G_1}(v_{j_1}) = d_{G_1}(f(v_1)), \quad d_{G_1}(v_2) = d_{G_1}(v_{j_2}) = d_{G_1}(f(v_2)). \quad (3)$$

In regard to Equation (3) and Figure 2, there exists the vertices v_{j_11} and v_{j_21} adjacent to vertices v_{j_1} and v_{j_2} , respectively. Thus the vertices v_{j_11} and v_{j_21} are adjacent to $G_2^{j_1}$ and $G_2^{j_2}$, respectively, and so $f^{-1}(v_{j_11}) \sim G_2^1$ and $f^{-1}(v_{j_21}) \sim G_2^2$. Hence $f^{-1}(v_{j_11}) \sim v_1$ and $f^{-1}(v_{j_21}) \sim v_2$. Since $v_{j_1} \sim v_{j_11}$ and $v_{j_2} \sim v_{j_21}$, so $f^{-1}(v_{j_1}) \sim f^{-1}(v_{j_11})$ and $f^{-1}(v_{j_2}) \sim f^{-1}(v_{j_21})$ (see Figure 3). Note that, for every vertex in $N_G(v_{j_2})$ such as x ,

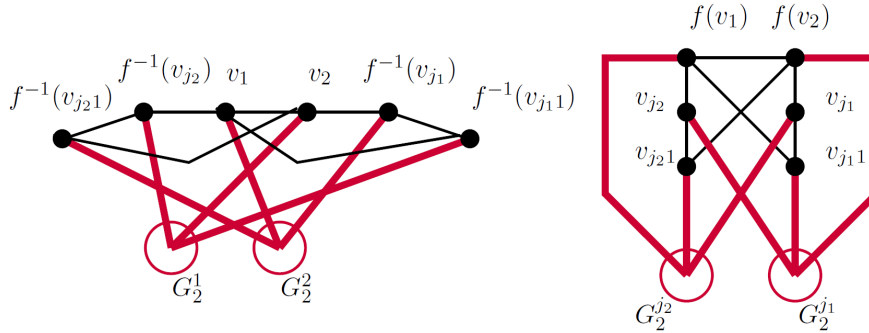


Figure 3: A piece of $G_1 \star G_2$ in the proof of Lemma 2.3.

we have $x \sim G_2^{j_2}$. So we see that $f^{-1}(x) \sim G_2^2$, and so $f^{-1}(x) \sim v_2$ (similar argument satisfies for each vertex in $N_G(v_{j_1})$). In regard to Figure 3 and Equation (3), we need to the other vertex adjacent to v_{j_1} , such as x . If x has been chosen among the nonadjacent vertices to $G_2^{j_2}$ that has been shown in Figure 3, then with the similar argument as above, we obtain that $f^{-1}(x)$ is adjacent to v_1 , and so Equation (3) dose not satisfy,

again. Therefore after finite steps we should choose a vertex adjacent to v_{j_1} , such as x , among the vertices that are adjacent to $G_2^{j_2}$, otherwise we conclude that the order of G_1 is infinite and this is a contradiction. By Figure 3 and above information, the vertex v_{j_1} is the only vertex that is adjacent to $G_2^{j_2}$ and is not among the adjacent vertices to v_{j_1} , in each step. Hence $v_{j_1} \sim v_{j_2}$, and the result follows. \square

Lemma 2.4 *Let G_1 and G_2 be two connected graphs of order n_1 and n_2 , respectively, and $n_1 > 1$. If f is an automorphism of $G_1 \star G_2$, then the restriction of f to G_1 is an automorphism of G_1 .*

Proof. Since f is an automorphism, it suffices to show that the restriction of f to G_1 is an automorphism of G_1 . By contradiction, suppose that $f|_{G_1}$ is not an automorphism of G_1 . Without loss of generality we assume that $f(v_1) = u_1^2$. Hence by Lemma 2.1, $d_{G_1}(v_2) > d_{G_1}(v_1)$. Since f preserves the degree of the vertices, $d_{G_1 \star G_2}(v_1) = d_{G_1 \star G_2}(u_1^2)$, and so by Equations (1) and (2) we have $(1 + n_2)d_{G_1}(v_1) = d_{G_2}(u_1) + d_{G_1}(v_2)$. Suppose that $N_{G_1}(v_1) = \{v_{1,1}, \dots, v_{1,s_1}\}$, $N_{G_1}(v_2) = \{v_{2,1}, \dots, v_{2,s_2}\}$ and $N_{G_2}(u_1) = \{u_{1,1}, \dots, u_{1,t}\}$ where $(1 + n_2)s_1 = t + s_2$ and $s_i = d_{G_1}(v_i)$, $i = 1, 2$, and also $t = d_{G_2}(u_1)$ (see Figure 4). Since f preserves the adjacency relation, so

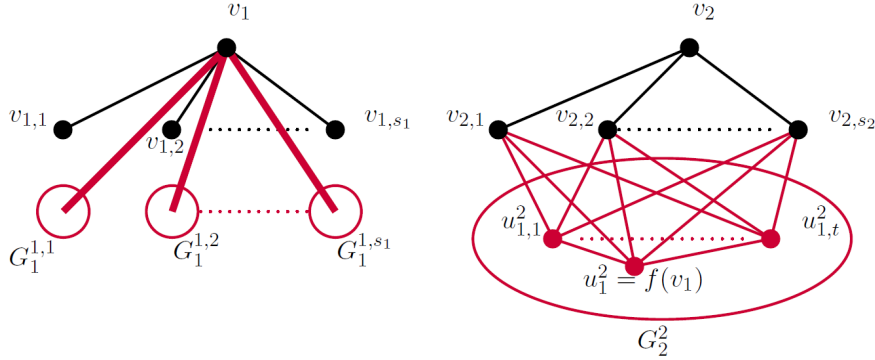


Figure 4: A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

$$f(N_{G_1 \star G_2}(v_1)) = N_{G_1 \star G_2}(u_1^2), \text{ i.e.,}$$

$$\begin{aligned} & \{f(v_{1,1}), \dots, f(v_{1,s_1}), f(u_{1,1}^{1,1}), \dots, f(u_{n_2}^{1,1}), \dots, f(u_{1,1}^{1,s_1}), \dots, f(u_{n_2}^{1,s_1})\} \\ &= \{u_{1,1}^2, \dots, u_{1,t}^2, v_{2,1}, \dots, v_{2,s_2}\}. \end{aligned} \quad (4)$$

Since $t < n_2$, there are vertices in the copies $G_2^{1,1}, \dots, G_2^{1,s_1}$ such that they are mapped to the elements of the set $\{v_{2,1}, \dots, v_{2,s_2}\}$, under the automorphism f . Without loss of generality we can assume that $f(u_{i_j}^{1,j}) = v_{2,j}$, where $1 \leq j \leq s_1$. We continue the proof by considering two cases for s_1 as follows:

Case 1) If $s_1 > 1$. Since v_2 is adjacent to the vertices $v_{2,1}, \dots, v_{2,s_1}$, so $f^{-1}(v_2)$ is adjacent to the vertices $u_{i_1}^{1,1}, \dots, u_{i_{s_1}}^{1,s_1}$. Since $s_1 > 1$, so $f^{-1}(v_2) \in G_1$ and $f^{-1}(v_2)$ is adjacent to the vertices $v_{1,1}, \dots, v_{1,s_1}$. Hence v_2 is adjacent to the vertices $f(v_{1,1}), \dots, f(v_{1,s_1})$,

and by Equation 4 we have

$$\{f(v_{1,1}), \dots, f(v_{1,s_1})\} \subseteq \{v_{2,s_1+1}, \dots, v_{2,s_2}\}. \quad (5)$$

Without loss of generality we assume that $f(v_{1,i}) = v_{2,s_1+i}$, where $1 \leq i \leq s_1$ (see Figure 5).

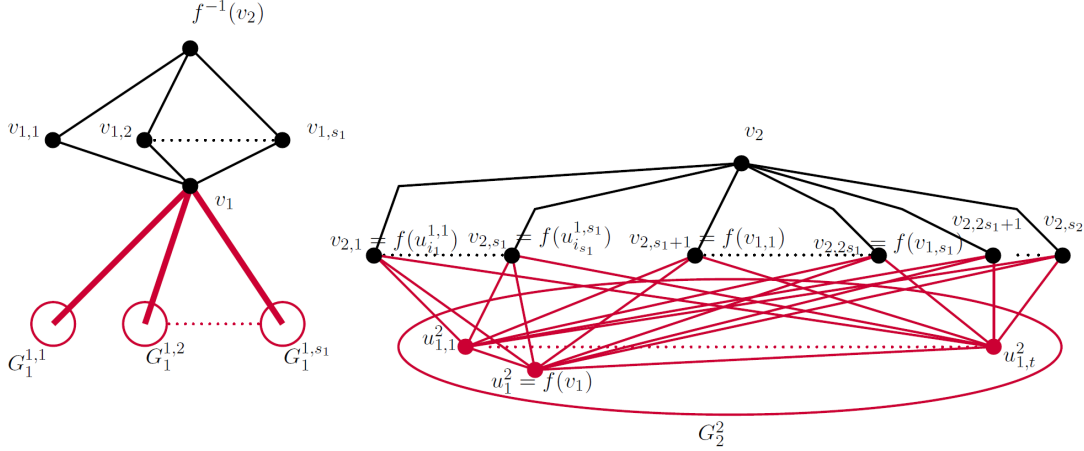


Figure 5: A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

Since $f^{-1}(v_2)$ is adjacent to the vertices $v_{1,1}, \dots, v_{1,s_1}$, we can say that $f^{-1}(v_2)$ is adjacent to all vertices of $G_2^{1,1}, \dots, G_2^{1,s_1}$, so v_2 is adjacent to all vertices of $f(G_2^{1,1}), \dots, f(G_2^{1,s_1})$. Then by Equation (4) we get

$$\{f(u_1^{1,1}), \dots, f(u_{n_2}^{1,1}), \dots, f(u_1^{1,s_1}), \dots, f(u_{n_2}^{1,s_1})\} \subseteq \{v_{2,1}, \dots, v_{2,s_2}\}. \quad (6)$$

With respect to Equations (4), (5) and (6) we have a contradiction.

Case 2) If $s_1 = 1$. Since f preserves the adjacency relation, so

$$\{f(v_{1,1}), f(u_1^{1,1}), \dots, f(u_{n_2}^{1,1})\} = \{u_{1,1}^2, \dots, u_{1,t}^2, v_{2,1}, \dots, v_{2,s_2}\}. \quad (7)$$

Since $t < n_2$, there exists a vertex in the copy $G_2^{1,1}$ such that it is mapped to an elements of the set $\{v_{2,1}, \dots, v_{2,s_2}\}$, under the automorphism f . Without loss of generality we can assume that $f(u_{i_1}^{1,1}) = v_{2,1}$. Since v_2 is adjacent to $v_{2,1}$, so $f^{-1}(v_2)$ is adjacent to $u_{i_1}^{1,1}$, and since $f^{-1}(v_2) \neq v_1$, so $f^{-1}(v_2) \in G_2^{1,1}$. Without loss of generality we can assume that $f^{-1}(v_2) = u_{i_1}^{1,1}$ such that $u_{i_1}^{1,1}$ is adjacent to $u_1^{1,1}$ (see Figure 6).

Since $v_{1,1}$ is adjacent to the vertex v_1 and $\text{dist}_{G_1 \star G_2}(v_{1,1}, u_{i_1}^{1,1}) = \text{dist}_{G_1 \star G_2}(v_{1,1}, u_{i_1}^{1,1}) = 2$, so $f(v_{1,1})$ is adjacent to the vertex u_1^2 and also

$$\text{dist}_{G_1 \star G_2}(f(v_{1,1}), v_2) = \text{dist}_{G_1 \star G_2}(f(v_{1,1}), v_{2,1}) = 2. \quad (8)$$

Now by Equations (7) and (8) we have a contradiction. Therefore the restriction of each automorphism of $G_1 \star G_2$ to G_1 is an automorphism of G_1 . \square

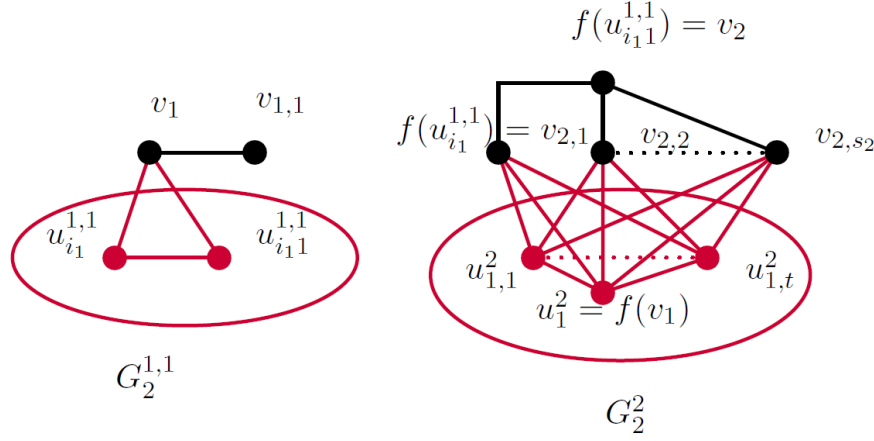


Figure 6: A piece of neighbourhood corona of G_1 and G_2 in the proof of Lemma 2.4.

Corollary 2.5 *Let G_1 and G_2 be two connected graphs of order n_1 and n_2 , respectively, such that $n_1 > 1$ and f is an automorphism of $G_1 \star G_2$. Then the restriction of f to G_1 is an automorphism of G_1 and also there are the automorphism g of G_1 and the automorphisms h_1, \dots, h_{n_1} of G_2 such that $f(G_2^i) = (h_i(G_2))^k$, where $v_k = g(v_i)$ and $i, k = 1, \dots, n_1$.*

Proof. By Lemmas 2.3 and 2.4, it is sufficient to prove that the copies of G_2 are mapped to each other under the automorphism f , and it follows from that f preserves the adjacency relation on each copy of G_2 . \square

The following corollary is an immediate consequence of Corollary 2.5 for graphs of the form $G \star K_1$.

Corollary 2.6 *Let G be a connected graph of order $n > 1$ and f be an arbitrary automorphism of $G \star K_1$. Then the restriction of f to G is an automorphism of G . Also $f(K_1^i) = K_1^{j_i}$ for some automorphism g of G such that $g(v_i) = v_{j_i}$ where $i, j_i = 1, 2, \dots, n_1$.*

3 Study of $D(G_1 \star G_2)$ and $D'(G_1 \star G_2)$

In this section we use the results in Section 2 to study the distinguishing number and the distinguishing index of the neighbourhood corona of two graphs. First we consider the neighbourhood corona of an arbitrary graph with K_1 . The following theorem gives an upper bound for $D(G \star K_1)$ and $D'(G \star K_1)$.

Theorem 3.1 *Let G be a connected graph of order $n > 1$. We have*

- (i) $D(G \star K_1) \leq D(G)$,

(ii) $D'(G \star K_1) \leq D'(G)$.

Proof.

- (i) We shall define a distinguishing vertex labeling for $G \star K_1$ with $D(G)$ labels. First we label G in a distinguishing way with $D(G)$ labels. Next we assign the vertex $K_1^{v_i}$, the label of the vertex v_i where $1 \leq i \leq n$. This labeling is a distinguishing vertex labeling of $G \star K_1$, because if f is an automorphism of $G \star K_1$ preserving the labeling then by Corollary 2.5, the restriction of f to G is an automorphism of G preserving the labeling. Since we labeled G in a distinguishing way at first, so the restriction of f to G is the identity automorphism on G . On the other hand by Corollary 2.6 there exists an automorphism g of G such that $f(K_1^{v_i}) = K_1^{g(v_i)}$, $1 \leq i \leq n$. Regarding to the labeling of copies of K_1 , we can obtain that g is the identity automorphism on G , and so f is the identity automorphism on $G \star K_1$.
- (ii) We define a distinguishing edge labeling for $G \star K_1$ with $D'(G)$ labels. First we label the edges of G in a distinguishing way with $D'(G)$ labels. By Equations (1) and (2) we know that the degree of $K_1^{v_i}$ in $G \star K_1$ is equal with the degree of v_i in G where $1 \leq i \leq n$. Now we assign the edge between $K_1^{v_i}$ and $v_{i,j}$ where $v_{i,j} \in N_G(v_i)$, the label of the edges between v_i and $v_{i,j}$ where $j = 1, \dots, d_G(v_i)$. This labeling is a distinguishing edge labeling of $G \star K_1$, because if f is an automorphism of $G \star K_1$ preserving the labeling then by Corollary 2.5, the restriction of f to G is an automorphism of G preserving the labeling. Since we labeled G in a distinguishing way at first, so the restriction of f to G is the identity automorphism on G . On the other hand by Corollary 2.6 there exists an automorphism g of G such that $f(K_1^{v_i}) = K_1^{g(v_i)}$, $1 \leq i \leq n$. Regarding to the labeling of the edges incident to each copies of K_1 , we can obtain that g is the identity automorphism on G , and so f is the identity automorphism on $G \star K_1$. \square

The bounds of $D(G \star K_1)$ and $D'(G \star K_1)$ in Theorem 3.1 are sharp. If we consider G as the star graph $K_{1,n}$, $n > 1$, then $K_{1,n} \star K_1$ is a graph as shown in Figure 7. Using the degree of the verices of $K_{1,n} \star K_1$ we can get the automorphism group of $K_{1,n} \star K_1$ and then it can be concluded that $D(K_{1,n} \star K_1) = n = D(K_{1,n})$, and also $D'(K_{1,n} \star K_1) = n = D'(K_{1,n})$.

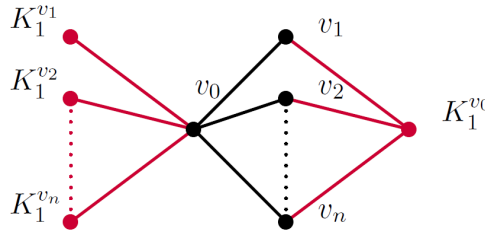


Figure 7: The neighbourhood corona of $K_{1,n}$ and K_1 .

In Theorem 3.1, the sharp upper bounds for $D(G \star K_1)$ and $D'(G \star K_1)$ have been given, but we did not present lower bounds for these parameters. Actually, there are graphs whose distinguishing number can be arbitrarily larger than the distinguishing number of its neighbourhood corona with K_1 . In other words, we can show that there exists a connected graph G of order $n > 1$ such that the value of $\frac{D(G \star K_1)}{D(G)}$ can be arbitrarily small. To do this we need the two following theorems. Recall that the friendship graph F_n ($n \geq 2$) can be constructed by joining n copies of the cycle graph C_3 with a common vertex.

Theorem 3.2 [2] *The distinguishing number of the friendship graph F_n ($n \geq 2$) is*

$$D(F_n) = \lceil \frac{1 + \sqrt{8n + 1}}{2} \rceil.$$

Now we obtain the exact value of the distinguishing number of neighborhood corona of F_n with K_1 .

Theorem 3.3 *The distinguishing number of $F_n \star K_1$ ($n \geq 2$) is*

$$D(F_n \star K_1) = \lceil \sqrt{\frac{1 + \sqrt{8n + 1}}{2}} \rceil.$$

Proof. Let $V(F_n) = \{v_0, v_1, v_2, \dots, v_{2n-1}, v_{2n}\}$ and the vertex v_0 be the central vertex and v_{2i-1} and v_{2i} be the vertices of the base of triangles in F_n where $1 \leq i \leq n$. So $d_{F_n}(v_0) = 2n$ and $d_{F_n}(v_i) = 2$ where $1 \leq i \leq 2n$. By Equations (1) and (2) we have $d_{F_n \star K_1}(v_0) = 4n$ and $d_{F_n \star K_1}(v_i) = 4$, also $d_{F_n \star K_1}(K_1^{v_0}) = 2n$ and $d_{F_n \star K_1}(K_1^{v_i}) = 2$ where $1 \leq i \leq 2n$ (see Figure 8).

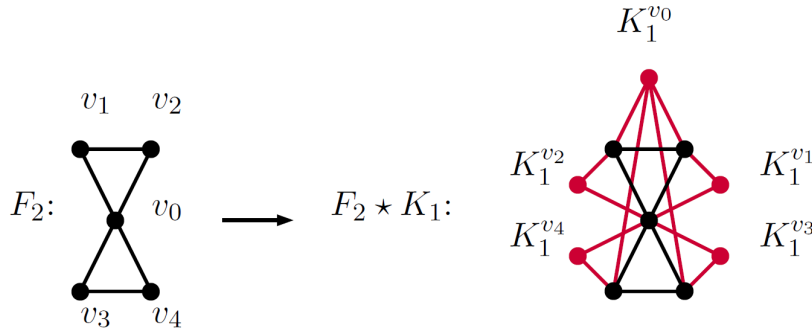


Figure 8: The graphs F_2 and $F_2 \star K_1$.

If f is an automorphism of $F_n \star K_1$, then f fixes the vertices v_0 and $K_1^{v_0}$ (if $n = 2$ we can get the same result by Corollary 2.5). So we assign the vertices v_0 and $K_1^{v_0}$ the label 1. Let (x_i, y_i, z_i, w_i) be the label of the vertices $(K_1^{v_{2i}}, v_{2i-1}, v_{2i}, K_1^{v_{2i-1}})$ where $1 \leq i \leq n$. Suppose that $L = \{(x_i, y_i, z_i, w_i) \mid 1 \leq i \leq n, x_i, y_i, z_i, w_i \in \mathbb{N}\}$, is a labeling of the vertices of $F_n \star K_1$ except the vertices v_0 and $K_1^{v_0}$. If L is a distinguishing labeling of $F_n \star K_1$ then:

- (i) For every $i = 1, \dots, n$, it should be satisfied that $x_i \neq w_i$ or $y_i \neq z_i$. Otherwise, the automorphism f_i of $F_n \star K_1$ such that f_i maps $K_1^{v_{2i}}$ and $K_1^{v_{2i-1}}$ to each other, two vertices v_{2i-1} and v_{2i} to each other, and fixes the remaining vertices, preserves the labeling.
- (ii) For every i and j in $\{1, \dots, n\}$, with $i \neq j$, it should be satisfied that $(x_i, y_i, z_i, w_i) \neq (x_j, y_j, z_j, w_j)$ and $(x_i, y_i, z_i, w_i) \neq (w_j, z_j, y_j, x_j)$. Otherwise, the automorphism f_{ij} and g_{ij} of $F_n \star K_1$ by the following definitions preserve the labeling.
 - The automorphism f_{ij} maps $K_1^{v_{2i}}$ and $K_1^{v_{2j}}$ to each other and also $K_1^{v_{2i-1}}$ and $K_1^{v_{2j-1}}$ to each other. The map f_{ij} maps v_{2i} and v_{2j} to each other, also it maps v_{2i-1} and v_{2j-1} to each other and fixes the remaining vertices of $F_n \star K_1$.
 - The automorphism g_{ij} maps $K_1^{v_{2i}}$ and $K_1^{v_{2j-1}}$ to each other, also $K_1^{v_{2i-1}}$ and $K_1^{v_{2j}}$ to each other. The map g_{ij} maps v_{2i} and v_{2j-1} to each other, also it maps v_{2i-1} and v_{2j} to each other and fixes the remaining vertices of $F_n \star K_1$.

So using the label set $\{1, \dots, s\}$ we can make at most $(s^4 - s^2)/2$ of the 4-ary's (x, y, z, w) satisfying (i) and (ii). Because, the number of 4-ary's (x, y, z, w) such that $x \neq w$ is $s(s-1)s^2$, and the number of 4-ary's (x, y, z, w) such that $y \neq z$ is $s(s-1)s^2$. On the other hand the number of 4-ary's (x, y, z, w) such that $x \neq w$ and $y \neq z$ is $(s(s-1))^2$. So the maximum number of 4-ary's (x, y, z, w) satisfying (i) is

$$(s(s-1)s^2 + s(s-1)s^2) - (s(s-1))^2 = s^4 - s^2.$$

Among these 4-ary's we should choose the 4-ary's that satisfying (ii), too. Therefore the number of 4-ary's (x, y, z, w) satisfying (i) and (ii) which they can make by the label set $\{1, \dots, s\}$ is $\frac{s^4 - s^2}{2}$. Therefore $D(F_n \star K_1) \geq \min\{s : \frac{s^4 - s^2}{2} \geq n\}$. By an easy computation, we see that

$$\min\{s : \frac{s^4 - s^2}{2} \geq n\} = \lceil \sqrt{\frac{1 + \sqrt{8n + 1}}{2}} \rceil.$$

Now we present a distinguishing vertex labeling with this number of labels. We assign v_0 and $K_1^{v_0}$ the label 1. We should label the remaining vertices such that the identity automorphism preserves the labeling only. Denoting each pentagon with the vertices $K_1^{v_{2i}}, v_{2i-1}, v_{2i}, K_1^{v_{2i-1}}, v_0$ in $F_n \star K_1$ where $1 \leq i \leq n$, by a general pentagon that have shown in Figure 9 and calling it a blade and continue the labeling. At first, we want to know the maximum number of blades that can be labeled in a distinguishing way by 1 and 2. As we can see in Figure 10, the maximum number of blades that can be labeled in distinguishing way, by 1, 2 is 6.

In order to preserve the labeling under the identity automorphism only, we should use another label to assign the next blade. As mentioned earlier, the maximum number of blades that can be labeled by each the set $\{1, 3\}, \{2, 3\}$ is six. Now we want to know the maximum number of blades that can be labeled by presence of $\{1, 2, 3\}$ at the same time in the blade. This number is 18. Because let to label with the labels 1, 2, 3 and

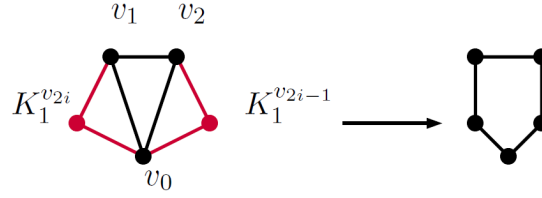


Figure 9: The considered pentagon (or a cycle of size 5) in the proof of Theorem 3.3.

a repetition of 1. As shown in Figure 10, we can label six blades. Obviously we can do the same with letting repetition of 2 and 3. Therefore the maximum number of blades that can be labeled by presence of $\{1, 2, 3\}$ at the same time is 18. Until now, we labeled 36 blades.

$$\underbrace{6}_{\{1,2\}} + \underbrace{6}_{\{1,3\}} + \underbrace{6}_{\{2,3\}} + \underbrace{18}_{\{1,2,3\}} = 36$$

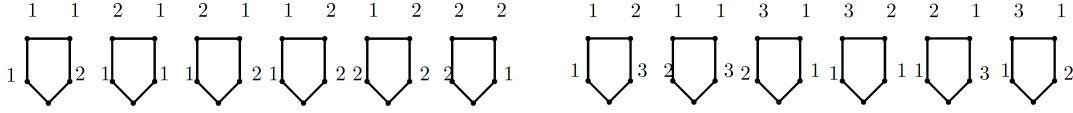


Figure 10: Distinguishing labeling of blades with the labels $\{1, 2\}$ and $\{1, 2, 3\}$, respectively.

If we want to label the next blade, we should add a new label, 4. The maximum number of blades that can be labeled by each the set $\{1, 4\}, \{2, 4\}, \{3, 4\}$ is six. Also, the maximum number of blades that can be labeled by each the set $\{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ is eighteen. We can see that the maximum number of blades that can be labeled by presence of $\{1, 2, 3, 4\}$ at the same time is 12 as Shown in Figure 11.

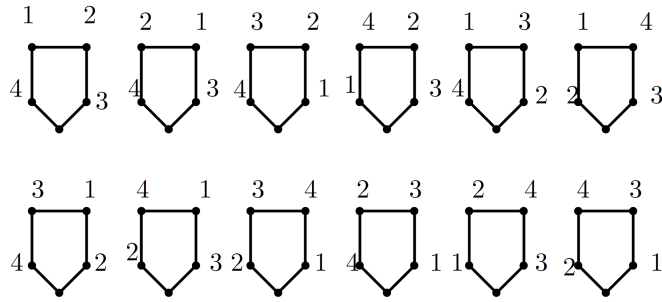


Figure 11: The distinguishing labeling of blades with the labels $\{1, 2, 3, 4\}$.

Thus we have labeled 120 blades until now.

$$36 + \underbrace{6}_{\{1,4\}} + \underbrace{6}_{\{2,4\}} + \underbrace{6}_{\{3,4\}} + \underbrace{18}_{\{1,2,4\}} + \underbrace{18}_{\{1,3,4\}} + \underbrace{18}_{\{2,3,4\}} + \underbrace{12}_{\{1,2,3,4\}} = 120.$$

Therefore the relationship between the number of labels that has been used, $\mathbf{d}(F_n \star K_1)$, and n are as the following sequence:

$$\{\mathbf{d}(F_n \star K_1)\} = \{0, \underbrace{2}_{6\text{-times}}, \underbrace{3}_{30\text{-times}}, \underbrace{4}_{84\text{-times}}, \dots, m, \dots, m, \dots\}.$$

where the number of the repetitions m in above sequence is $(m-1)6 + \binom{m-1}{2}18 + \binom{m-1}{3}12$, with $m \geq 1$.

In fact, $\mathbf{d}(F_n \star K_1) = \min\{k : \sum_{i=1}^k \left(\binom{i-1}{1}6 + \binom{m-1}{2}18 + \binom{m-1}{3}12 \right) \geq n\}$. By an easy computation, we see that

$$\begin{aligned} \min\{k : \sum_{i=1}^k \left(\binom{i-1}{1}6 + \binom{m-1}{2}18 + \binom{m-1}{3}12 \right) \geq n\} \\ = \min\{k : (k^4 - k^2)/2 \geq n\} \\ = \lceil \sqrt{\frac{1 + \sqrt{8n+1}}{2}} \rceil. \end{aligned}$$

Therefore we have the result. \square

Now we are ready to state and prove the following theorem:

Theorem 3.4 *There exists a connected graph G of order $n > 1$ such that the value of $\frac{D(G \star K_1)}{D(G)}$ can be arbitrarily small.*

Proof. By Theorems 3.2 and 3.3 it can be seen that

$$\lim_{n \rightarrow \infty} \frac{D(F_n \star K_1)}{D(F_n)} = \lim_{n \rightarrow \infty} \frac{\lceil \sqrt{\frac{1 + \sqrt{8n+1}}{2}} \rceil}{\lceil \frac{1 + \sqrt{8n+1}}{2} \rceil} = 0$$

Therefore we have the result. \square

The following theorem is one of the main result of this paper and gives an upper bound for the distinguishing number of the neighbourhood corona of two arbitrary graphs:

Theorem 3.5 *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2 , respectively, such that $n_1 > 1$. Then $D(G_1 \star G_2) \leq \max\{D(G_1), D(G_2) + M\}$, where*

$$M = \min \left\{ k : \sum_{m=0}^k y_m \geq D(G_1) \right\}, \quad y_m = \begin{cases} 1 & m = 0, \\ D(G_2) & m = 1, \\ D(G_2) + \sum_{i=1}^{m-1} \binom{m-1}{i} \binom{D(G_2)}{i+1} & m \geq 2. \end{cases}$$

Proof. We define a distinguishing vertex labeling for $G_1 \star G_2$ with $\max\{D(G_1), D(G_2) + M\}$ labels. First we label G_1 with $D(G_1)$ labels in a distinguishing way. For the labeling of copies of G_2 , we partition the vertices of G_1 by the distinguishing labeling of G_1 , i.e., we partition the vertices of G_1 into $D(G_1)$ classes, such that $[i]$ th class contains the vertices of G_1 having the label i , in the distinguishing labeling of G_1 , where $1 \leq i \leq D(G_1)$. Let $[i] = \{v_{i1}, \dots, v_{is_i}\}$, where s_i is the size of $[i]$ th class and $1 \leq i \leq D(G_1)$. By this partition we label the copies of G_2 as follows: First we label the vertices of G_2 with $D(G_2)$ labels in a distinguishing way, next we do the following changes on the labeling of G_2 . Before the labeling of the copies of G_2 , we introduce the notation $G_2^{[i]}$ for the set $\{G_2^{i1}, \dots, G_2^{is_i}\}$, i.e., $G_2^{[i]}$ is the set of that copies of G_2 corresponding to the elements of $[i]$ th class, where $1 \leq i \leq D(G_1)$. In fact we partition the copies of G_2 into $D(G_1)$ classes, that $G_2^{[i]}$ is the notation of $[i]$ th class. Now we present the labeling of copies of G_2 by the following steps:

Step 1) We label all of the copies of G_2 which are in $G_2^{[1]}$, exactly the same as the distinguishing labeling of G_2 .

Step 2) For the labeling of the copies in $G_2^{[i]}$, where $2 \leq i \leq D(G_2) + 1$, we use of the new label $D(G_2) + 1$ in such a way that the label $i - 1$ in the all elements of $G_2^{[i]}$ is replaced by the new label $D(G_2) + 1$, where $2 \leq i \leq D(G_2) + 1$.

Step 3) For the labeling of the copies in $G_2^{[i]}$, where $D(G_2) + 2 \leq i \leq 2D(G_2) + 1$, we do the same action as Step 2, with the new label $D(G_2) + 2$, instead of the labels $D(G_2) + 1$.

Step 4) By choosing two labels among the labels $\{1, \dots, D(G_2)\}$, and replacing them by the two new labels $D(G_2) + 1$ and $D(G_2) + 2$, we can label the elements of $\binom{D(G_2)}{2}$ other classes of the classes $G_2^{[i]}$.

Step 5) We do the same work as Step 2 with the new label $D(G_2) + 3$ instead of labels $D(G_2) + 1$. Next we label $2\binom{D(G_2)}{2}$ other classes $G_2^{[i]}$, with the two new labels $D(G_2) + 1$ and $D(G_2) + 3$, also with the labels $D(G_2) + 2$ and $D(G_2) + 3$, exactly the same as Step 4.

Step 6) Now we choose three labels among the labels $\{1, \dots, D(G_2)\}$, and replace them by the three new labels $D(G_2) + 1$, $D(G_2) + 2$ and $D(G_2) + 3$.

By continuing this method we conclude that the number of classes can be labeled with the labels $1, \dots, D(G_2) + m$, $m \geq 1$, such that the label $D(G_2) + m$ is used in the labeling of each element of classes, is y_m where

$$y_m = \begin{cases} 1 & m = 0, \\ D(G_2) & m = 1, \\ D(G_2) + \sum_{i=1}^{m-1} \binom{m-1}{i} \binom{D(G_2)}{i+1} & m \geq 2. \end{cases}$$

Therefore the number of labels that have been used for the labeling of all copies of G_2 , is $D(G_2) + M$ where $M = \min \left\{ k : \sum_{m=0}^k y_m \geq D(G_1) \right\}$. This labeling is a distinguishing vertex labeling of $G_1 \star G_2$, because if f is an automorphism of $G_1 \star G_2$ preserving the labeling, then by Corollary 2.5, $f|_{G_1}$ is an automorphism of G_1 preserving the labeling. Since we labeled G_1 in a distinguishing way, at first, so f is the identity

automorphism on G_1 . Regarding to the labeling of copies of G_2 and since f preserves the labeling of the copies of G_2 , so f maps each copy of G_2 to itself. The map f is the identity automorphism on each copy of G_2 , because each copy of G_2 was labeled in a distinguishing way. Therefore f is the identity automorphism on $G_1 \star G_2$. \square

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6 *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2 , respectively, such that $n_1 > 1$. If $D(G_1) = 1$, then $D(G_1 \star G_2) \leq D(G_2)$.*

Proof. It is sufficient to note that if $D(G_1) = 1$, then the value of M in Theorem 3.5 is zero. \square

We end the paper by presenting an upper bound for the distinguishing index of the neighbourhood corona of two graphs:

Theorem 3.7 *Let G_1 and G_2 be two connected graphs of orders n_1 and n_2 , respectively, such that $n_1 > 1$. Then $D'(G_1 \star G_2) \leq \max\{D'(G_1), D'(G_2)\}$.*

Proof. We define an edge distinguishing labeling of $G_1 \star G_2$ with $\max\{D'(G_1), D'(G_2)\}$ labels. To obtain such labeling we first label the edge set of G_1 and G_2 in a distinguishing way with $D'(G_1)$ and $D'(G_2)$ labels, respectively. For the labeling of the edges between each copy of G_2 and G_1 we use of the labeling of the edge set of G_1 as follows:

Let $N_{G_1}(v_k) = \{v_{k1}, \dots, v_{|N_{G_1}(v_k)|}\}$, where $1 \leq k \leq n_1$. By the notations of the vertices of G_1 and the copies of G_2 , we assign the all edges $v_{kj_k}u_i^k$, $1 \leq i \leq n_2$, the label of the edge $v_{kj_k}v_k$ in the distinguishing labeling of the edge set of G_1 , where $1 \leq k \leq n_1$ and $1 \leq j_k \leq |N_{G_1}(v_k)|$. This labeling is a distinguishing edge labeling of $G_1 \star G_2$, because if f is an automorphism of $G_1 \star G_2$ preserving the labeling, then by Corollary 2.5, the restriction of f to G_1 is an automorphism of G_1 preserving the labeling. Since we labeled G_1 in a distinguishing way, at first, so f is the identity automorphism on G_1 . Regarding to the labeling of the edges between the copies of G_2 and G_1 and by Corollary 2.5 we conclude that f maps each copy of G_2 to itself. Since we labeled each copy of G_2 in a distinguishing way, at first, so the map f is the identity automorphism on each copy of G_2 , and so f is the identity automorphism on $G_1 \star G_2$. \square

References

- [1] M.O. Albertson and K.L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin. 3 (1996) #R18.
- [2] S. Alikhani and S. Soltani, *Distinguishing number and distinguishing index of certain graphs*, submitted. Available at <http://arxiv.org/abs/1602.03302>.
- [3] R. Frucht and F. Harary, *On the corona two graphs*, Aequationes Math. 4 (1970) 322-325.

- [4] F. Harary, *Graph Theory*, Addition-Wesley Publishing Co., Reading, MA/Menlo Park, CA/London, 1969.
- [5] I. Gopalapillai, *The spectrum of neighborhood corona of graphs*, Kragujevac Journal of Mathematics. 35 (2011) 493-500.
- [6] R. Kalinowski and M. Pilsniak, *Distinguishing graphs by edge colourings*, European J. Combin. 45 (2015) 124-131.
- [7] S. Klářzar and X. Zhu, *Cartesian powers of graphs can be distinguished by two labels*, European J. Combin. 28 (2007) 303-310.
- [8] X. Liu and S. Zhou, *Spectra of the neighbourhood corona of two graphs*, Linear Multilinear Alg. 62, 9 (2014) 1205–1219.
- [9] F. Michael and I. Garth, *Distinguishing colorings of Cartesian products of complete graphs*, Discrete Math., 308 (11), (2008) 2240-2246.
- [10] E. Sampathkumar, H. B. Walikar, *On the splitting graph of a graph*, Karnatak Univ. J. Sci. 35/36 (1980-1981), 13-16.